MATHEMATICS Proceedings A 89 (3), September 29, 1986

How to construct extensional combinatory algebras

by Ingemarie Bethke

Mathematical Institute, University of Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, the Netherlands

Communicated by Prof. A.S. Troelstra at the meeting of February 24, 1986

ABSTRACT

We develop a slight modification of Engeler's graph algebras, yielding extensional combinatory algebras. It is shown that by this construction we get precisely the class of Scott's D_∞-models generated by complete atomic Boolean algebras. In section 3 we construct extensional substructures of graph-algebras and $P\omega$-models.

0. INTRODUCTION

0.1 DEFINITION

(i) A *combinatory algebra* (*ca*) is a structure $(A, *, K, S)$ with $*$ a binary operation ('application') on A and two distinguished elements $K, S \in A$ satisfying

$$AK \quad \forall x \in A\, \forall y \in A\ Kxy = x$$

$$AS \quad \forall x \in A\, \forall y \in A\, \forall z \in A\ Sxyz = xz(yz)$$

where xy is short for $x * y$.

(ii) Moreover, such a structure is *extensional* iff

$$EXT \quad \forall x \in A\, \forall y \in A\ (\forall z \in A\ xz = yz \to x = y) \quad \square$$

In [E] Engeler introduced the notion of a graph algebra over an arbitrary non-empty set. The construction starts with a non-empty carrier set A. Then $G(A)$ is the least set containing A such that for $b \in G(A)$ and finite $B \subseteq G(A)$ the pair (B, b) is in $G(A)$, assuming that all $a \in A$ are not such pairs, that is

0.2 DEFINITION. Let $A \neq \emptyset$ and $G(A) := \bigcup \{G_n(A) | n \in \omega\}$ where $G_n(A)$ is recursively defined by

(i) $G_0(A) := A$

(ii) $G_{n+1}(A) := G_n(A) \cup \{(B, b) | B \subseteq G_n(A),\ B \text{ finite},\ b \in G_n(A)\}$. $\square$

A binary application operation $\bullet$ on the subsets of $G(A)$ is then defined by

$$X \bullet Y := \{b | \exists B \subseteq Y((B, b) \in X)\}.$$

Engeler showed that the graph algebra $(P(G(A)), \bullet)$ over a non-empty set A can be made into a *ca* by isolating appropriate subsets K and S of $G(A)$. These structures are very elegant, since the notion of application is easy to grasp: the result of applying X to Y depends on the 'elementary instructions' (B, b) of X, which give output b any time the input Y contains B. Since this construction never yields extensional *ca*'s, we shall give:

1. A SLIGHTLY MODIFIED CONSTRUCTION FOR EXTENSIONAL *CA*'S

Again we start with an arbitrary non-empty set A. In the description below we let small letters $a, b, c, \ldots, x, y, z$ range over $G(A)$ and capital letters $B, C, \ldots, X, Y, Z$ denote subsets of $G(A)$. On $P(G(A))$ we define an application operation by

1.1 DEFINITION. $X * Y := \{b | \exists B \leq Y \ (B, b) \in X\} \cup \{a \in A | a \in X\}$ where we put $Z \leq Z' \leftrightarrow \forall x \in Z \exists y \in Z'(x \leq_{G(A)} y)$ and $x \leq_{G(A)} y$ holds if either

(i) $x = y$ or

(ii) $\exists B \exists b\ (x = (B, b)\ \&\ y \in A\ \&\ b \leq_{G(A)} y)$ or

(iii) $\exists b\ (x \in A\ \&\ y = (\emptyset, b)\ \&\ x \leq_{G(A)} b)$ or

(iv) $\exists B_1 \exists B_2 \exists b_1 \exists b_2\ (x = (B_1, b_1)\ \&\ y = (B_2, b_2)\ \&\ B_2 \leq B_1\ \&\ b_1 \leq_{G(A)} b_2)$ $\square$

REMARKS. Observe that for all X $A \bullet X = \emptyset = \emptyset \bullet X$, since we have assumed A not to contain pairs of the form (B, b). So, if we want to construct an extensional *ca*, while leaving the application operation unchanged, we are forced to identify A with $\emptyset$, which would have unpleasant consequences. Therefore we consider the elements of A also to be elementary instructions needing no input at all and producing themselves. Moreover, for all X $\{(\emptyset, b)\} \bullet X = \{(\emptyset, b), (B, b)\} \bullet X$. Hence $\{(\emptyset, b)\}$ and $\{(\emptyset, b), (B, b)\}$ represent the same function and should therefore be considered as being equal. On the other hand there is always a subset of $G(A)$ which separates $\{(\emptyset, b)\}$ from $\{(\emptyset, b), (B, b)\}$ if $B \neq \emptyset$: for example, let $D := \{(\{(B, b)\}, b)\}$. Then $D \bullet \{(\emptyset, b)\} = \emptyset$ but $D \bullet \{(\emptyset, b), (B, b)\} = \{b\}$. Therefore we change normal set theoretical inclusion into a relation $Z \leq Z'$ which may be read as "Z' contains at least as strong instructions as Z". y is at least as strong as x ($x \leq_{G(A)} y$) is then defined by 4 clauses:

either x and y denote the same instruction (i)

or y needs no input to produce an output which is at least as strong as the output of x (ii, iii) or y needs at most as much input as x to produce an output at least as strong as the output of x (iv).

1.2 PROPOSITION

(i) $\forall x \forall y\ (x=y \to x \leq_{G(A)} y)$

(ii) $\forall X \forall Y\ (X \subseteq Y \to X \leq Y)$

(iii) $\forall x \in A\ \forall y \in A\ (x=y \leftrightarrow x \leq_{G(A)} y)$

(iv) $\forall X \subseteq A\ \forall Y \subseteq A\ (X \subseteq Y \leftrightarrow X \leq Y)$

(v) $\forall x \exists a \in A\ (x \leq_{G(A)} a)$

(vi) $\forall X\ (\emptyset \leq X \leq A)$

PROOF. Easy. □

$\leq_{G(A)}$ and $\leq$ are transitive relations on $G(A)$ and $P(G(A))$ respectively:

1.3 PROPOSITION

(i) $\forall a \in A\ \forall y\ (a \leq_{G(A)} y \leftrightarrow (\emptyset, a) \leq_{G(A)} y)$

(ii) $\forall a \in A\ \forall y\ (y \leq_{G(A)} a \leftrightarrow y \leq_{G(A)} (\emptyset, a))$

(iii) $\leq_{G(A)}$ is transitive

(iv) $\leq$ is transitive

PROOF. (i) Let $a \in A$ and y be arbitrary. Then $a \leq_{G(A)} y \leftrightarrow$

$$a=y \text{ or } \exists c\ (y=(\emptyset, c)\ \&\ a \leq_{G(A)} y) \leftrightarrow$$

$$(y \in A\ \&\ a \leq_{G(A)} y) \text{ or } \exists c\ (y=(\emptyset, c)\ \&\ a \leq_{G(A)} y) \leftrightarrow (\emptyset, a) \leq_{G(A)} y.$$

(ii) similar. (iii) We prove with induction on n:

$$\forall n \in \omega\ \forall x \in G_n(A)\ \forall y \in G_n(A)\ \forall z \in G_n(A)\ (x \leq_{G(A)} y\ \&\ y \leq_{G(A)} z \to x \leq_{G(A)} z).$$

The transitivity of $\leq_{G(A)}$ then follows from the observation that for all $\{x, y, z\} \subseteq G(A)$ there is an $n \in \omega$ with $\{x, y, z\} \subseteq G_n(A)$. If $\{x, y, z\} \subseteq G_0(A)$ are such that $x \leq_{G(A)} y$, $x \leq_{G(A)} y$ then $x=y=z$. Hence $x \leq_{G(A)} z$. Suppose $\{x, y, z\} \subseteq G_{n+1}(A)$ are such that $x \leq_{G(A)} y$, $x \leq_{G(A)} y$. Define

$$S(u) := \begin{cases} (\emptyset, u) \text{ if } u \in A \\ u \quad \text{otherwise} \end{cases}$$

Then from (i), (ii) we get $S(x) \leq_{G(A)} S(y)$, $S(y) \leq_{G(A)} S(z)$. Hence there are finite $B_i \subseteq G_n(A)$, $b_i \in G_n(A)$ for $i=1,2,3$ such that $S(x)=(B_1, b_1)$, $S(y)=(B_2, b_2)$, $S(z)=(B_3, b_3)$, $B_3 \leq B_2 \leq B_1$ and $b_1 \leq_{G(A)} b_2 \leq_{G(A)} b_3$. So from the induction hypothesis it follows that $B_3 \leq B_1$ and $b_1 \leq_{G(A)} b_3$. Thus $S(x) \leq_{G(A)} S(z)$. Then again from (i), (ii) we get $x \leq_{G(A)} z$. (iv) follows immediately from (iii). □

1.4 DEFINITION. Let $A \neq \emptyset$. Define $M(A) := (P(G(A))/\equiv, *, [K], [S])$, where

$$X \equiv Y \leftrightarrow X \leq Y \;\&\; Y \leq X$$

$$[X] := \{Y \mid Y \equiv X\} \text{ and } P(G(A))/\equiv \; := \{[X] \mid X \in P(G(A))\}$$

$$[X] * [Y] := [X * Y]$$

$$K := \{(B, (C, b)) \mid b \in B\}$$

$$S := \{(B, (C, (D, b))) \mid \exists U(\exists D' \leq D \; (D', (U, b)) \in B \;\&$$

$$\& \; \forall u \in U \exists u' \geq_{G(A)} u \exists D' \leq D \; (D', u') \in C)\} \quad \square$$

$\equiv$ is by definition symmetric, reflexive by 1.2(ii) and transitive by 1.3(iii). Hence $\equiv$ is an equivalence relation.

1.5 PROPOSITION
(i) $\forall X \forall Y \; (X = Y \rightarrow X \equiv Y)$
(ii) $\forall X \in P(A) \forall Y \in P(A) \; (X = Y \leftrightarrow X \equiv Y)$
(iii) $A \equiv G(A)$

PROOF. Easy. $\square$

Before we show that $M(A)$ is an extensional *ca* we prove

1.6 LEMMA (MONOTONICITY). $*$ is monotone wrt. $\leq$, i.e.
(i) $\forall X \forall Y \forall Z \; (X \leq Y \rightarrow ZX \leq ZY)$
(ii) $\forall X \forall Y \forall Z \; (X \leq Y \rightarrow XZ \leq YZ)$

PROOF. (i) Suppose $X \leq Y$ and let $b \in ZX$. Then if $b \in \{a \in A \mid a \in Z\}$ also $b \in ZY$. If $b \notin \{a \in A \mid a \in Z\}$ then $(B, b) \in Z$ for some $B \leq X$. Thus since $X \leq Y$ also $B \leq Y$. Hence again $b \in ZY$. (ii) Suppose $X \leq Y$ and let $b \in XZ$. If $b \in \{a \in A \mid a \in X\}$ then $b \leq_{G(A)} b'$ for some $b' \in Y$. Hence either $b' \in A$ or $b' = (\emptyset, b'')$ with $b \leq_{G(A)} b''$. So there is a $d \in YZ$ with $b \leq_{G(A)} d$. If $b \notin \{a \in A \mid a \in X\}$ then $(B, b) \in X$ for some $B \leq Z$. Let $b' \in Y$ be such that $(B, b) \leq_{G(A)} b'$. If $b' \in A$ then $b \leq_{G(A)} b'$ and $b' \in YZ$. If $b' \notin A$ then $b' = (D, d)$ with $D \leq B \leq Z$, $b \leq_{G(A)} d$ and $d \in YZ$. $\square$

1.7 LEMMA (EXTENSIONALITY)
(i) $\forall X \forall Y \; (\forall B \text{ finite } (XB \leq YB) \rightarrow X \leq Y)$
(ii) $\forall X \forall Y \; (\forall B \text{ finite } (XB \equiv YB) \rightarrow X \equiv Y)$
(iii) $\forall X \forall Y \; (\forall Z \; (XZ \equiv YZ) \rightarrow X \equiv Y)$

PROOF. (i) Suppose for all finite B $XB \leq YB$ and let $b \in X$. If $b \in A$ then $b \in X\emptyset \leq Y\emptyset$. Let $b' \in Y\emptyset$ be such that $b \leq_{G(A)} b'$. Then either $b' \in \{a \in A \mid a \in Y\}$ or $(\emptyset, b') \in Y$. So $b \leq_{G(A)} b''$ for some $b'' \in Y$. If $b \notin A$ then $b = (C, c)$ and $c \in XC \leq YC$. Hence $c \leq_{G(A)} c'$ for some $c' \in YC$. Then either $c' \in \{a \in A \mid a \in Y\}$

or $(D,c')\in Y$ for some $D\leq C$. Thus again $b\leq_{G(A)}b''$ for some $b''\in Y$. (ii) and (iii) follow from (i) and (ii) respectively. $\square$

1.8 THEOREM. Let $A\neq\emptyset$. Then $M(A)$ is a *ca* satisfying extensionality.

PROOF. $P(G(A))/\equiv$ is clearly closed under $*$ and from the monotonicity of $*$ it follows that $\forall X\forall Y\forall Z([X]=[Y]\to[X][Z]=[Y][Z]\ \&\ [Z][X]=[Z][Y])$. So $*$ is a binary operation on $P(G(A))/\equiv$. To prove AK let X, Y be arbitrary. Then

$$KXY=\{(C,b)\mid\exists B\leq X\ b\in B\}Y$$
$$=\{(C,b)\mid\exists b'\in X(b\leq_{G(A)}b')\}Y=\{b\mid\exists b'\in X(b\leq_{G(A)}b')\}\equiv X.$$

Thus $\forall X\forall Y\,[K][X][Y]=[X]$. To prove AS let X, Y, Z be arbitrary and choose $X'\equiv X$ and $Y'\equiv Y$ such that all $x\in X'$ are of the form $(D,(U,b))$ and all $y\in Y'$ of the form (D,b). Then $[S][X][Y][Z]=[S][X'][Y'][Z]$ and $[X][Z]([Y][Z])=$ $=[X'][Z]([Y'][Z])$. We will show $[S][X'][Y'][Z]=[X'][Z]([Y'][Z])$, i.e. $SX'Y'Z\equiv X'Z(Y'Z)$. Now $X'Z=\{(U,b)\mid\exists D\leq Z((D,(U,b))\in X')\}$ and $Y'Z=$ $=\{b\mid\exists D\leq Z((D,b)\in Y')\}$. Hence

$$\begin{aligned}X'Z(Y'Z)&=\{b\mid\exists U\leq Y'Z((U,b)\in X'Z)\}=\\&=\{b\mid\exists U\leq Y'Z\exists D\leq Z((D,(U,b))\in X')\}\\&=\{b\mid\exists U(\exists D\leq Z((D,(U,b))\in X')\ \&\\&\quad\&\ \forall u\in U\exists u'\geq u\exists D\leq Z((D,u)\in Y))\}.\end{aligned}$$

On the other hand

$$\begin{aligned}SX'Y'Z&=\{b\mid\exists D\leq Z\exists C\leq Y'\exists B\leq X'\exists U(\exists D'\leq D((D',(U,b))\in B)\ \&\\&\quad\&\ \forall u\in U\exists u'\geq_{G(A)}u\exists D'\leq D((D',u)\in C))\}=\\&=\{b\mid\exists b'\geq_{G(A)}b\exists U(\exists D\leq Z((D,(U,b'))\in X')\ \&\\&\quad\&\ \forall u\in U\exists u'\geq_{G(A)}u\exists D\leq Z((D,u)\in Y))\}.\end{aligned}$$

Thus $SX'Y'Z\equiv X'Z(Y'Z)$. Finally, we have to prove $[K]\neq[S]$. As is well known, it suffices to show $[X]\neq[Y]$ for some $[X],[Y]\in P(G(A))/\equiv$. So let $a\in A$ $(A\neq\emptyset!)$. Then $[\{a\}],[\{(\{a\},a)\}]\in P(G(A))/\equiv$ and $\neg\ \{a\}\leq\{(\{a\},a)\}$. Hence $[\{a\}]\neq[\{(\{a\},a)\}]$. Thus $M(A)$ is a *ca* and by lemma 1.7(iii) $M(A)$ satisfies extensionality. $\square$

2. THE GLOBAL STRUCTURE OF $M(A)$

Clearly all $M(A)$'s are up to isomorphism uniquely determined by the cardinality of their carrier set A, but we also have the converse, i.e. $\forall A\forall A'\ (M(A)\cong M(A')\to\text{Card }(A)=\text{Card }(A))$. Before we prove this fact we will state several properties of $M(A)$. Notice that $(M(A),\leq)$ where

$$[X]\leq[Y]\leftrightarrow X\leq Y$$

is a complete lattice with bottom $[\emptyset]$, top $[A]$ and the supremum $\sup F = [\bigcup\{X \mid [X] \in F\}]$ for arbitrary $F \subseteq P(G(A))/\equiv$. We will show that some of the lattice theoretic properties of $M(A)$ can be expressed in $M(A)$. First we will prove that the binary SUP resp. INF operator is definable (in the language of *ca*'s plus constants) in $M(A)$.

2.1 LEMMA. There is a $\mathrm{SUP}_A \in M(A)$ such that

(i) $\forall X \forall Y\ ([X] \leq \mathrm{SUP}_A[X][Y]\ \&\ [Y] \leq \mathrm{SUP}_A[X][Y])$

(ii) $\forall X \forall Y \forall Z\ ([X] \leq [Z]\ \&\ [Y] \leq [Z] \to \mathrm{SUP}_A[X][Y] \leq [Z])$

(iii) $\forall Z\ (\forall X \forall Y\ ([Z][X][\emptyset] = [X] = [Z][X][X]\ \&\ [Z][X][Y] = [Z][Y][X]) \to$
$\to [Z] = \mathrm{SUP}_A)$

PROOF. Define $\mathrm{SUP}_A := [\{(B,(C,d)) \mid \{d\} \leq B \cup C\}]$.

Then $\mathrm{SUP}_A[X][Y] = [\{d \mid \{d\} \leq X \cup Y\}]$. From this (i) follows immediately. To prove (ii) suppose $[X] \leq [Z]$, $[Y] \leq [Z]$. Then from the monotonicity it follows that $\mathrm{SUP}_A[X][Y] \leq \mathrm{SUP}_A[Z][Z] = [\{d \mid \{d\} \leq Z\}] = [Z]$. Hence $\mathrm{SUP}_A[X][Y] \leq [Z]$. (iii) Let Z be arbitrary such that $\forall X \forall Y\ ([Z][X][\emptyset] = [X] = [Z][X][X]\ \&\ [Z][X][Y] = [Z][Y][X])$. Then $[X] = [Z][X][\emptyset] \leq [Z][X][Y]$ and $[Y] = [Z][Y][\emptyset] \leq [Z][Y][X] = [Z][X][Y]$. Thus for arbitrary X, Y we get from (ii) $\mathrm{SUP}_A[X][Y] \leq [Z][X][Y]$. Moreover, $[Z][X][Y] \leq [Z](\mathrm{SUP}_A[X][Y])(\mathrm{SUP}_A[X][Y]) = \mathrm{SUP}_A[X][Y]$. Hence for all X, Y $[Z][X][Y] = \mathrm{SUP}_A[X][Y]$. So $[Z] = \mathrm{SUP}_A$ by extensionality. □

2.2 LEMMA. There is a $\mathrm{INF}_A \in M(A)$ such that

(i) $\forall X \forall Y\ (\mathrm{INF}_A[X][Y] \leq [X]\ \&\ \mathrm{INF}_A[X][Y] \leq [Y])$

(ii) $\forall X \forall Y \forall Z\ ([Z] \leq [X]\ \&\ [Z] \leq [Y] \to [Z] \leq \mathrm{INF}_A[X][Y])$

(iii) $\forall Z\ (\forall X \forall Y\ ([Z][X][A] = [X] = [Z][X][X]\ \&$
$\&\ [Z][X][Y] = [Z][Y][X]) \to [Z] = \mathrm{INF}_A)$

PROOF. Define $\mathrm{INF}_A = [\{(B,(C,d)) \mid \{d\} \leq B\ \&\ \{d\} \leq C\}]$. □

Next we characterize top and bottom:

2.3 LEMMA.

$$\forall X\ [\exists Y \exists Z\ ([Y] \neq [X]\ \&\ [Z][X] = [X]\ \&\ \forall X' \not\equiv X [Z][X'] = [Y])$$

$$\leftrightarrow ([X] = [\emptyset] \text{ or } ([X] = [A]\ \&\ A \text{ is finite})]$$

PROOF. $\to$: Let $[Y] \neq [X], [Z]$ be such that $[Z][X] = [X]$ and $\forall X' \not\equiv X$ $[Z][X'] = [Y]$. Suppose $[X] \neq [\emptyset]$ and $[X] \neq [A]$. Then since $[\emptyset] \leq [X] \leq [A]$ we get $[Y] = [Z][\emptyset] \leq [Z][X] = [X] = [Z][X] \leq [Z][A] = [Y]$, i.e. $[X] = [Y]$. Contradiction. Thus $[X] = [\emptyset]$ or $[X] = [A]$. Suppose $[X] \neq [\emptyset]$ and A is infinite. We will show $A \equiv Y$. Clearly $Y \leq A$. To prove $A \leq Y$ let $a \in A$ be arbitrary. Then since $A \equiv ZA$ $a \leq_{G(A)} b$ for some $b \in ZA$. If $b \in \{a \in A \mid a \in Z\}$ then $b \in Z\emptyset \equiv Y$. Hence $a \leq_{G(A)} b \leq_{G(A)} b'$ for some $b' \in Y$. If $b \notin \{a \in A \mid a \in Z\}$ then $(B, b) \in Z$ for some finite $B \leq A$. Now since B is finite $B \not\equiv A$. So $b \in ZB \equiv Y$. Thus again $a \leq_{G(A)} b \leq_{G(A)} b'$ for some $b' \in Y$. So $A \equiv Y$, i.e. $[A] = [Y]$. Contradiction. Hence A is finite. $\leftarrow$: For $[\emptyset]$ choose $a_0 \in A$ and define $[Z] := [\{(B, a_0) | B \neq \emptyset\}]$. Then $[Z][\emptyset] = [\emptyset]$ and $\forall X' \not\equiv \emptyset$ $[Z][X'] = [\{a_0\}] \neq [\emptyset]$. If A is finite then choose $a_0 \in A$ and define $[Z] := [\{(A, a) | a \in A\} \cup \{(\emptyset, (\{a_0\}, a_0))\}]$. Then $[Z][A] = [A \cup \{(\{a_0\}, a_0)\}] = [A]$. Moreover, if $[X'] \neq [A]$ then $\neg A \leq X'$. So $[Z][X'] = [\{(\{a_0\}, a_0)\}] \neq [A]$ for all $[X'] \neq [A]$. □

Observe that $(\{[X] | X \in P(A)\}, \leq)$ is a complete sublattice of $(M(A), \leq)$ which is isomorphic to $(P(A), \subseteq)$ by proposition 1.2(vi). We will finally show that the elements of this sublattice are definable in $M(A)$.

2.4 LEMMA. $\forall Z\, (\forall X\, ([Z][X] = [X]) \leftrightarrow \exists Y \in P(A)\, ([Z] = [Y]))$

PROOF. $\rightarrow$: Suppose $\forall X$ $[Z][X] = [X]$ and define $Y := \{a \in A | \exists X\, (X \equiv Z \;\&\; a \in X)\}$. Clearly $[Y] \leq [Z]$. To prove $[Z] \leq [Y]$ let $b \in Z$. If $b \in A$ then $b \in Y$. If $b \notin A$ then $b = (B_1, \ldots (B_n, a) \ldots)$ for some $a \in A$ and finite B_i. So $b \leq_{G(A)} a \in ZB_1 \ldots B_n \equiv Z$. Thus $b \leq_{G(A)} a \in Y$. $\leftarrow$: Suppose $[Z] = [Y]$ for some $Y \in P(A)$. Then for all X $[Z][X] = [Y][X] = [YX] = [Y] = [Z]$. □

Now we are ready to prove

2.5 THEOREM. $\forall A \forall A'\, (M(A) \cong M(A') \leftrightarrow \text{Card}\ (A) = \text{Card}\ (A'))$

PROOF. $\rightarrow$: Suppose $M(A) \cong M(A')$ by some bijection $\theta : P(G(A_1))/\!\equiv\; \rightarrow P(G(A_2))/\!\equiv$ such that $\forall X \forall Y\, \theta([X][Y]) = \theta([X])\theta([Y])$. Let $B \in P(A)$. Then for all $[X] \in P(G(A))/\!\equiv$ $[B][X] = [B]$. So for all $[X] \in P(G(A'))/\!\equiv$ $\theta([B])[X] = \theta([B])$ and thus by lemma 2.4 $\theta([B]) = [C]$ for some $C \in P(A')$. By the same argument we also see that for all $C \in P(A')$ we have $[C] = \theta([B])$ for some $B \in P(A)$. So $\theta[\{[X] | X \in P(A)\}] = \{[X] | X \in P(A')\}$. Hence Card $(\{[X] | X \in P(A)\})$ = Card $(\{[X] | X \in P(A')\})$ and thus by proposition 1.5(ii) Card $(P(A))$ = Card $(P(A'))$. Then if A' is finite Card (A) = Card (A'). Suppose A' is infinite. We shall prove that $(\{[X] | X \in P(A)\}, \leq) \cong (\{[X] | X \in P(A')\}, \leq)$. By lemma 2.3 there are $[Y] \neq [\emptyset]$ and $[Z] \in P(G(A))/\!\equiv$ such that $[Z][\emptyset] = [\emptyset]$ and for all $X' \not\equiv \emptyset$ $[Z][X'] = [Y]$. Then again by lemma 2.3 $\theta([\emptyset]) = [\emptyset]$. Now observe that for all $[X], [Y] \in P(G(A))/\!\equiv$ we have $\text{SUP}_A[X][\emptyset] = [X] = \text{SUP}_A[X][X]$ and $\text{SUP}_A[X][Y] = \text{SUP}_A[Y][X]$. Thus for all $[X], [Y] \in P(G(A'))/\!\equiv$ $\theta(\text{SUP}_A)[X][\emptyset] = [X] = \theta(\text{SUP}_A)[X][X]$ and $\theta(\text{SUP}_A)[X][Y] = \theta(\text{SUP}_A)[Y][X]$. So by lemma 2.1(iii) $\theta(\text{SUP}_A) = \text{SUP}_{A'}$. Hence for all $B, C \in P(A)$

$$[B] \leq [C] \leftrightarrow \text{SUP}_A[B][C] = [C] \leftrightarrow \text{SUP}_{A'}\theta([B])\theta([C]) = \theta([C]) \leftrightarrow$$

$$\leftrightarrow \theta([B]) \leq \theta([C]).$$

So $(\{[X] | X \in P(A)\}, \leq) \cong (\{[X] | X \in P(A')\}, \leq)$. Hence $(P(A), \subseteq) \cong (P(A'), \subseteq)$ and Card (A) = Card (A') follows immediately. $\leftarrow$: Easy and left to the reader. □

Let $(D_1, \leq_1)$, $(D_2, \leq_2)$ be complete partial orders (cpo), then $[D_1 \to D_2]$ is the set of continuous maps considered as a cpo by pointwise ordering. It is well known that every cpo $(D, \leq')$ with $(D, \leq') \cong [D \to D]$ by some continuous bijection can be made into an extensional *ca*. In [S1], [S2] Scott showed how to construct complete lattices $(D_\infty, \leq_\infty) \cong [D_\infty \to D_\infty]$ starting with an arbitrary complete lattice $(D_0, \leq_0)$. This construction can be done also for cpo's rather than complete lattices. Now, the question arises whether we get different extensional *ca*'s by the graph construction. The answer is no. We shall show that for every $A \neq \emptyset$ and every cpo $(D_0, \leq_0)$ we have

$$M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty) \leftrightarrow (P(A), \subseteq) \cong (D_0, \leq_0).$$

Hence the graph construction yields up to isomorphism exactly those *ca*'s $(D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D_0, \leq_0)$ a complete atomic Boolean algebra. We shall first give a brief outline of the D_∞-construction and extract the properties of D_∞ we will need in the proofs below. For a very thorough discussion see [B].

Let $(D_0, \leq_0)$ be a cpo and define inductively $(D_{n+1}, \leq_{n+1}) := [D_n \to D_n]$. Then $(D_\infty, \leq_\infty)$ is the cpo with

$D_\infty := \{\langle x_0, x_1, \dots \rangle \mid \forall n \in \omega (x_n \in D_n \;\&\; \varphi_n(x_{n+1}) = x_n\}$ for certain fixed projections $\varphi_n \in [D_{n+1} \to D_n]$

$\langle \vec{x} \rangle \leq_\infty \langle \vec{y} \rangle$ iff $\forall n \in \omega \; x_n \leq_n y_n$

$\sup F := \langle \sup \{x_n | x \in F\} \rangle_{n \in \omega}$ for directed $F \subseteq D_\infty$

$\perp_\infty := \langle \perp_n \rangle_{n \in \omega}$, where $\perp_n$ is the bottom of $(D_n, \leq_n)$.

Furthermore, there are projections $P_n \in [D_\infty \to D_\infty]$, such that $(P_n[D_\infty], \leq_\infty)$ is a sub-cpo of $(D_\infty, \leq_\infty)$, and isomorphisms $\mathrm{fun}_n \in [P_n[D_\infty] \to D_n]$. We will abbreviate $P_n(x)$ with x_n. Then the following laws of projection hold in D_∞: $\forall x \in D_\infty \, \forall n \in \omega \, \forall m \in \omega$

(L1) $(x_n)_m = x_{\min\{n,m\}}$

(L2) $n \leq m \to x_n \leq_\infty x_m \leq_\infty x$

(L3) $x = \sup \{x_n | n \in \omega\}$

(L4) $\perp_n = \perp_\infty$

(L5) $(\sup F)_n = \sup \{x_n | x \in F\}$ for arbitrary subsets $F \subseteq D_\infty$ with existing sup's.

Moreover, a binary application operation $*_\infty$ is defined on D_∞ which satisfies the below laws of application: $\forall x \in D_\infty \, \forall y \in D_\infty \, \forall n \in \omega \, \forall m \in \omega$

(L6) $x_{n+1} *_\infty y = x_{n+1} *_\infty y_n = (x *_\infty y_n)_n$

(L7) $x_0 *_\infty y = x_0 = (x *_\infty \perp_\infty)_0$

(L8) $(\sup F) *_\infty x = \sup \{y *_\infty x | y \in F\}$ for arbitrary $F \subseteq D_\infty$ with existing sup's

(L9) $x *_\infty \sup F = \sup \{x *_\infty y \mid y \in F\}$ for directed $F \subseteq D_\infty$

(L10) $\forall z\ (y \leq_\infty z \rightarrow x *_\infty y \leq_\infty x *_\infty z \ \&\ y *_\infty x \leq_\infty z *_\infty x)$ (monotonicity)

$$\left.\begin{array}{ll}\text{(L11)} & \forall z\ x *_\infty z \leq_\infty y *_\infty z \rightarrow x \leq_\infty y \\ \text{(L12)} & \forall z\ x *_\infty z = y *_\infty z \rightarrow x = y\end{array}\right\}\ \text{(extensionality)}$$

(L13) $\mathrm{fun}_n(x_{n+1} *_\infty y_n) = \mathrm{fun}_{n+1}(x_{n+1})(\mathrm{fun}_n(y_n))$.

We turn our attention now to the particular D_∞-models which are generated by algebraic lattices.

2.6 DEFINITION. Let $(D, \leq')$ be a complete lattice. Then
(i) $x \in D$ is *compact* iff for every $X \subseteq D$ one has $x \leq' \sup X \rightarrow x \leq' \sup Y$ for some finite $Y \subseteq X$.
(ii) $C(D) := \{x \in D \mid x \text{ is compact}\}$
(iii) $(D, \leq')$ is *algebraic* iff for all $x \in D$ $x = \sup \{y \mid y \leq' x \ \&\ y \in C(D)\}$ □

The structure of an algebraic lattice is completely determined by the dense subset $C(D)$. Therefore we now characterize $C(D_\infty)$. It is easy to see that if $(D_0, \leq_0)$ is an algebraic lattice, then all $(D_n, \leq_n)$ and $(D_\infty, \leq_\infty)$ are algebraic lattices. In the sequel we assume $(D_0, \leq_0)$ to be an algebraic lattice.

2.7 PROPOSITION
(i) $\forall n \in \omega\, \forall m \in \omega\ (n \leq m \rightarrow C(P_n[D_\infty]) \subseteq C(P_m[D_\infty])$
(ii) $C(D_\infty) = \bigcup \{C(P_n[D_\infty]) \mid n \in \omega\}$

PROOF. Easy and left to the reader. □

2.8 PROPOSITION. For all $n \in \omega$ and for all $y, y' \in C(P_n[D_\infty])$ there is a unique $x_{y,y'} \in C(P_{n+1}[D_\infty])$ such that for all $z \in D_\infty$

$$x_{y,y'} *_\infty z = \begin{cases} y' \text{ if } y \leq_\infty z \\ \perp_\infty \text{ otherwise} \end{cases} \qquad (\dagger)$$

PROOF. Let $n \in \omega$ and $y, y' \in C(P_n[D_\infty])$ be arbitrary.
Then $\mathrm{fun}_n(y), \mathrm{fun}_n(y') \in C(D_n)$.

Define $f: D_n \rightarrow D_n$ by

$$f(x) := \begin{cases} \mathrm{fun}_n(y') \text{ if } \mathrm{fun}_n(y) \leq_n x \\ \mathrm{fun}_n(\perp_n) \text{ otherwise} \end{cases}$$

Then $f \in [D_n \rightarrow D_n] = D_{n+1}$ and $f \in C(D_{n+1})$:
Suppose $f \leq_{n+1} \sup F$ for some $F \subseteq D_{n+1}$. Then

$$\mathrm{fun}_n(y') = f(\mathrm{fun}_n(y)) \leq_n \sup \{g(\mathrm{fun}_n(y)) \mid g \in F\}.$$

So since $\mathrm{fun}_n(y') \in C(D_n)$ there is a finite $F_0 \subseteq F$ such that

$$\mathrm{fun}_n(y') \leq_n \sup \{g(\mathrm{fun}_n(y)) | g \in F_0\}.$$

Thus for all $z \in D_n$ $f(z) \leq \sup \{g(z) | g \in F_0\}$, i.e. $f \leq_{n+1} \sup F_0$.

Now define $x_{y,y'} := \mathrm{fun}_{n+1}^{-1}(f)$. Then $x_{y,y'} \in C(P_{n+1}[D_\infty])$ and (†) follows from L1, L2, L4, L6 and L13. The uniqueness of $x_{y,y'}$ is due to the extensionality of D_∞. □

2.9 PROPOSITION. For all $n \in \omega$ and all $x \in D_\infty$

$$x \in C(P_{n+1}[D_\infty]) \leftrightarrow \exists m \in \omega \, \forall i \leq m \, \exists y_i \in C(P_n[D_\infty]) \exists z_i \in C(P_n[D_\infty])$$

$$x = \sup \{x_{y_i, z_i} | i \leq m\}$$

PROOF. Let $x \in C(P_{n+1}[D_\infty])$ and define $X := \{x_{y,z} | y, z \in C(P_n[D_\infty])$ & & $z \leq_\infty x *_\infty y\}$. Then $X \subseteq P_{n+1}[D_\infty]$ and $x = \sup X$. Hence there is a finite $X_0 \subseteq X$ such that $x = \sup X_0$, i.e. there is a $m \in \omega$ and $y_0, \ldots, y_m, z_0, \ldots, z_m \in C(P_n[D_\infty])$ such that $x = \sup \{x_{y_i, z_i} | i \leq m\}$. □

Before we prove the characterization theorem we shall show

$$(P(G(A)), / \equiv, *) \cong (D_\infty, *_\infty)$$

with $(D_0, \leq_0) = (P(A), \subseteq)$. As a first step in that direction we isolate a certain subset of D_∞ which corresponds to the set of 'elementary instructions' $G(A)$.

2.10 DEFINITION. Let $\mathrm{Elem}(D_\infty) := \bigcup \{\mathrm{Elem}_n(D_\infty) | n \in \omega\}$ where $\mathrm{Elem}_n(D_\infty)$ is recursively defined by

(i) $\mathrm{Elem}_0(D_\infty) := \{\mathrm{fun}_0^{-1}(\{a\}) | a \in A\}$

(ii) $\mathrm{Elem}_{n+1}(D_\infty) := \mathrm{Elem}_n(D_\infty) \cup \{x_{y,z} | \exists$ finite $X \subseteq \mathrm{Elem}_n(D_\infty)$ $y = \sup X$ & & $z \in \mathrm{Elem}_n(D_\infty)\}$ □

2.11 PROPOSITION

(i) $\forall x \in \mathrm{Elem}(D_\infty) \forall X \subseteq \mathrm{Elem}(D_\infty) \, (x \leq_\infty \sup X \rightarrow \exists x_0 \in X \; x \leq_\infty x_0)$

(ii) $\forall x \in C(D_\infty) \exists$ finite $X \subseteq \mathrm{Elem}(D_\infty)$ $x = \sup X$

(iii) $\forall x \in D_\infty$ $x = \sup \{y \in \mathrm{Elem}(D_\infty) | y \leq_\infty x\}$

PROOF. (i) With induction on $x \in \mathrm{Elem}_n(D_\infty)$. Let $x \in \mathrm{Elem}_0(D_\infty)$ and suppose $x \leq_\infty \sup X$ with $X \subseteq \mathrm{Elem}(D_\infty)$. Then for some $a \in A$

$$x = \mathrm{fun}_0^{-1}(\{a\}) \leq_\infty \sup X.$$

Hence $\mathrm{fun}_0^{-1}(\{a\}) \leq_\infty (\sup X)_0 = \sup \{x_0 | x \in X\}$ by L5. Thus

$$\{a\} \subseteq \bigcup \{\mathrm{fun}_0(x_0) | x \in X\},$$

i.e. $\{a\} \subseteq \mathrm{fun}_0(x_0)$ for some $x \in X$. Hence $\mathrm{fun}_0^{-1}(\{a\}) \leq_\infty x_0 \leq_\infty x$ by L2. Let $x_{y,z} \in \mathrm{Elem}_{n+1}(D_\infty)$ and suppose $x_{y,z} \leq_\infty \sup X$ for $X \subseteq \mathrm{Elem}(D_\infty)$. Then

$$z = x_{y,z} *_\infty y \leq_\infty \sup X *_\infty y = \sup \{x *_\infty y | x \in X\} =$$
$$= \sup \{\mathrm{fun}_0^{-1}(\{a\}), z | \mathrm{fun}_0^{-1}(\{a\}) \in X \text{ or } \exists y' \leq_\infty y\ x_{y',z} \in X\}$$
$$\subseteq \mathrm{Elem}(D_\infty).$$

Thus from the induction hypothesis it follows that $z \leq_\infty \mathrm{fun}_0^{-1}(\{a\}) \in X$ or $z \leq_\infty z'$ for some $x_{y',z'} \in X$ with $y' \leq_\infty y$. Then it follows from the monotonicity of $*_\infty$ that $x_{y,z} \leq_\infty x$ for some $x \in X$. (ii) With induction on $x \in C(P_n[D_\infty])$. If $x \in C(P_0[D_\infty])$ then for some finite $B \subseteq A$ $x = \sup \{\mathrm{fun}_0^{-1}(\{a\}) | a \in B\}$. Let $x \in C(P_{n+1}[D_\infty])$. Then by proposition 2.9 $x = \sup \{x_{y_i, z_i} | i \leq m\}$ for certain $m \in \omega$ and $y_i, z_i \in C(P_n[D_\infty])$. Thus from the induction hypothesis it follows that $y_i = \sup Y_i$, $z_i = \sup Z_i$ with finite $Y_i, Z_i \subseteq \mathrm{Elem}(D_\infty)$. So

$$x = \sup \{x_{y_i, z_i} | i \leq m\} = \sup \{x_{y_i, z} | z \in Z_i \ \&\ i \leq m\}$$

and

$$\{x_{y_i, z} | z \in Z_i \ \&\ i \leq m\} \subseteq \mathrm{Elem}(D_\infty).$$

(iii) follows from (ii) and the algebraic nature of $(D_\infty, \leq_\infty)$. □

2.12 DEFINITION. For $b \in G(A)$ define inductively
(i) $\varphi(a) := \mathrm{fun}_0^{-1}(\{a\})$ if $a \in A$
(ii) $\varphi(B, b) := x_{\sup \{\varphi(b) | b \in B\}, \varphi(b)}$ □

2.13 LEMMA
(i) $\forall b \in G(A)\ \varphi(b) \in \mathrm{Elem}(D_\infty)$
(ii) $\forall b \in G(A) \forall c \in G(A)\ (b \leq_{G(A)} c \leftrightarrow \varphi(b) \leq_{G(A)} \varphi(c))$
(iii) $\forall x \in \mathrm{Elem}(D_\infty) \exists b \in G(A)\ \varphi(b) = x$

PROOF. (i) and (iii) follow immediately from definition 2.12. For (ii) we prove with induction on n: $\forall n \in \omega \forall \{b, c\} \subseteq P(G_n(A))(b \leq_{G(A)} c \leftrightarrow \varphi(b) \leq_\infty \varphi(c))$. For $n = 0$ this is trivial. Let $\{b, c\} \subseteq P(G_{n+1}(A))$. Define

$$S(u) := \begin{cases} (\emptyset, u) \text{ if } u \in A \\ u \quad \text{otherwise} \end{cases}$$

Then if $u \in A$ $\varphi(u) = \mathrm{fun}_0^{-1}(\{u\})$ and $\varphi(S(u)) = x_{\perp_\infty, \mathrm{fun}_0^{-1}(\{u\})}$. So from proposition 2.8 and L7 it follows that $\varphi(u) = \varphi(S(u))$ for all $u \in G(A)$. Suppose $b \leq_{G(A)} c$. Then $S(b) \leq_{G(A)} S(c)$. So there are $D_i \subseteq G_n(A)$ and $d_i \in G_n(A)$ for $i = 1, 2$ such that $S(b) = (D_1, d_1)$, $S(c) = (D_2, d_2)$, $D_2 \leq D_1$ and $d_1 \leq_{G(A)} d_2$. Then we get from the induction hypothesis $\sup \{\varphi(d) | d \in D_2\} \leq_\infty \sup \{\varphi(d) | d \in D_1\}$ and $\varphi(d_1) \leq_\infty \varphi(d_2)$. Hence

$$\varphi(b) = \varphi(S(b)) = x_{\sup \{\varphi(d) | d \in D_1\}, \varphi(d_1)} \leq_\infty x_{\sup \{\varphi(d) | d \in D_2\}, \varphi(d_2)} = \varphi(S(c)) = \varphi(c).$$

Suppose $\varphi(b) \leq_\infty \varphi(c)$. Then $\varphi(D_1, d_1) \leq_\infty \varphi(D_2, d_2)$ with $S(b) = (D_1, d_1)$, $S(c) = = (D_2, d_2)$, $D_i \subseteq G_n(A)$ and $d_i \in G_n(A)$ for $i = 1, 2$. So

$$\varphi(d_1) = \varphi(S(b)) *_\infty \sup \{\varphi(d) | d \in D_1\} \leq_\infty \varphi(S(c)) *_\infty \sup \{\varphi(d) | d \in D_1\}.$$

Thus $d_1 \leq_\infty d_2$ and $D_2 \leq D_1$ by the induction hypothesis and proposition 2.11(i). □

Next we prove

2.14 THEOREM. $\forall A\ M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D_0, \leq_0) = (P(A), \subseteq)$.

PROOF. Define $\theta : P(G(A))/\equiv \to D_\infty$ by $\theta([X]) := \sup \{\varphi(b) | b \in X\}$. Then θ is a monotone bijection by lemma 2.12 and proposition 2.11(iii). Moreover for all $[X], [Y] \in P(G(A))/\equiv$

$$\theta([X][Y]) = \theta([XY]) = \sup \{\varphi(b) | b \in XY\} =$$
$$= \sup \{\varphi(b) | b \in \{b' | \exists B \leq Y (B, b) \in X\} \cup \{a \in A | a \in X\}\} =$$
$$\sup \{\mathrm{fun}_0^{-1}(\{a\}), z | (a \in A \ \&\ a \in X) \text{ or}$$
$$\exists [B] \leq [Y] \exists b\ (\varphi(b) = z \ \&\ (B, b) \in X)\} =$$
$$= \sup \{\mathrm{fun}_0^{-1}(\{a\}), z | (a \in A \ \&\ a \in X) \text{ or}$$
$$\exists y \leq_\infty \theta([Y])\ x_{y,z} \in \{\varphi(b) | b \in X\}\} = \sup \{\varphi(b) *_\infty \theta([Y]) | b \in X\} =$$
$$= \sup \{\varphi(b) | b \in X\} *_\infty \theta([Y]) = \theta([X]) *_\infty \theta([Y]).$$

Finally, since D_∞ is extensional $\theta([K]) = K_\infty$ and $\theta([S]) = S_\infty$. □

Now we are ready to prove

2.15 CHARACTERIZATION THEOREM. For all $A \neq \emptyset$ and for all cpo's $(D_0 \leq_0)$

$$M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty) \leftrightarrow (P(A), \subseteq) \cong (D_0, \leq_0).$$

PROOF. $\to$: Suppose $M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ via some bijection

$$\theta : P(G(A))/\equiv \to D_\infty$$

such that for all $[X], [Y] \in P(G(A))/\equiv$ $\theta([X][Y]) = \theta([X]) *_\infty \theta([Y])$. Then by lemma 2.4 for all $X \in P(A)$ and $x \in D_\infty$ $\theta([X]) *_\infty x = \theta([X])$. Since in D_∞ we have

$$\forall x \in D_\infty (x = x_0 \leftrightarrow \forall y \in D_\infty x *_\infty y = x)$$

(cf. [8], 18.4.18) we get $\forall X \in P(A)$ $\theta([X]) = (\theta([X]))_0$. Thus $\theta' : P(A) \to D_0$ defined by $\theta'(X) := \mathrm{fun}_0(\theta([X]))$ is a bijection. Now, since $A \neq \emptyset$ $\mathrm{Card}(D_0) \geq 2$. Choose $d_0 \in D_0$ with $d_0 \neq \perp_0$ and define $f : D_0 \to D_0$ by

$$f(x) := \begin{cases} d_0 & \text{if } x \neq \perp_0 \\ \perp_0 & \text{otherwise} \end{cases}$$

Then $f \in D_1$. Hence there is an $y \in D_\infty$ such that for all $z \in D_\infty$ we have

$$y *_\infty z = \begin{cases} \mathrm{fun}_0^{-1}(d_0) \text{ if } z \neq \bot_\infty \\ \bot_\infty \quad \text{otherwise.} \end{cases}$$

Thus by lemma 2.3 $\theta^{-1}(\bot_\infty) = [\emptyset]$ or $\theta^{-1}(\bot_\infty) = [A]$.

Assume $\theta([A]) = \bot_\infty$. Then from lemma 2.2 it follows that for all $x, y \in D_\infty$ $\theta(\mathrm{INF}_A)x\bot_\infty = x = \theta(\mathrm{INF}_A)xx$ and $\theta(\mathrm{INF}_A)xy = \theta(\mathrm{INF}_A)yx$. Then it is easy to see that for all $x, y \in D_\infty \theta(\mathrm{INF}_A)xy = \sup\{x, y\}$. So for all $[X], [Y] \in P(A)$ we have

$$X \subseteq Y \leftrightarrow \mathrm{INF}_A[X][Y] = [X] \leftrightarrow \theta(\mathrm{INF}_A)\theta([X])\theta([Y]) = \theta([X]) \leftrightarrow$$
$$\leftrightarrow \theta([Y]) \leq_\infty \theta([X]) \leftrightarrow \theta'(Y) \leq_0 \theta'(X).$$

Hence $(P(A), \subseteq) \cong (D_0, \leq_0)$ via $\theta'': P(A) \to D_0$ defined by $\theta''(X) = \theta'(A \setminus X)$.

Assume $\theta(\emptyset) = \bot_\infty$. Then similarly we see that $\theta(\mathrm{SUP}_A)xy = \sup\{x, y\}$. So for all $X, Y \in P(A)$ we have

$$X \subseteq Y \leftrightarrow \mathrm{SUP}_A[X][Y] = [Y] \leftrightarrow \theta(\mathrm{SUP}_A)\theta([X])\theta([Y]) = \theta([Y]) \leftrightarrow$$
$$\leftrightarrow \theta([X]) \leq_\infty \theta([Y]) \leftrightarrow \theta'(X) \leq_0 \theta'(Y).$$

Thus again $(P(A), \subseteq) \cong (D_0, \leq_0)$. $\leftarrow$: If $(P(A), \subseteq) \cong (D_0, \leq_0)$ then clearly $(D'_\infty, *_\infty, K_\infty, S_\infty) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D'_0, \leq_0) = (P(A), \subseteq)$. Hence from theorem 2.12 it follows that $M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$. □

3. EXTENSIONAL SUBSTRUCTURES OF $(P(G(A)), \bullet)$ AND $P\omega$

As already mentioned in the introduction Engeler's graph algebra $(P(G(A)), \bullet)$ is never extensional. However, there is always a substructure which can be made into an extensional *ca*, provided $A \neq \emptyset$. We will show this by embedding $(P(G(A))/\equiv, *)$ isomorphically into $(P(G(A)), \bullet)$.

3.1 THEOREM. $\forall A\ (P(G(A))/\equiv, *) \hookrightarrow (P(G(A)), \bullet)$.

PROOF. Define $\theta: P(G(A))/\equiv \to P(G(A))$ by $\theta([X]) := \bigcup\{Z \mid Z \equiv X\}$ and observe that $\bigcup\{Z \mid Z \equiv X\} \equiv X$. Then θ is an injection. Moreover,

$$\theta([X]) \bullet \theta([Y]) = \{b \mid \exists B \subseteq \theta([Y])\ (B, b) \in \theta([X])\} =$$
$$= \{b \mid \exists B \subseteq \bigcup\{Z \mid Z \equiv Y\}\ (B, b) \in \bigcup\{Z \mid Z \equiv X\}\} =$$
$$= \{b \mid \exists B \leq Y\ \{(B, b)\} \leq X\} = \bigcup\{Z \mid Z \equiv XY\} = \theta([XY]) = \theta([X][Y]).\ \square$$

As known from the literature the $P\omega^{c,e}$-models are non-extensional *ca*'s, whose structures depend on the specific codings c of pairs of natural numbers and e of finite subsets of ω used in the construction. Given two bijections $c: \omega^2 \to \omega$, $e: \omega \to \{X \in P(\omega) \mid X \text{ finite}\}$ $P\omega^{c,e}$ is the model $(P(\omega), \blacksquare)$ with the application on $P(\omega)$ defined by

$$X \blacksquare Y := \{m \in \omega \mid \exists n \in \omega (e(n) \subseteq Y \ \&\ c(n, m) \in X)\}.$$

In [S3] Scott presents a very elegant method to construct extensional substructures of the $P\omega^{c,e}$-models. Here we will give a more elementary technique by embedding $(P(G(A_{c,e}))/\equiv, *)$ isomorphically into $P\omega^{c,e}$ for a certain set $A_{c,e}$. However, for rather 'nice' codings only this technique will yield non-trivial extensional substructures.

3.2 DEFINITION. Let $c:\omega^2\to\omega$, $e:\omega\to\{X\in P(\omega)|X \text{ finite}\}$ be arbitrary.

(i) Define $A_{c,e} := \{n\in\omega | c(e^{-1}(\emptyset), n) = n\}$

(ii) For $b\in G(A_{c,e})$ define inductively

$$\varphi(n) := n \text{ if } n\in A_{c,e}$$

$$\varphi(B,b) := c(e^{-1}(\{\varphi(b)|b\in B\}), \varphi(b))$$

(iii) Define $\theta: P(G(A_{c,e}))/\equiv \to\omega$ by $\theta([X]) := \{\varphi(b)|b\in\bigcup\{Z|Z\equiv X\}\}$ □

3.3 LEMMA. $\forall [X]\in P(G(A_{c,e}))/\equiv \forall [Y]\in P(G(A_{c,e}))/\equiv$

(i) $\theta([X]) = \theta([Y]) \leftrightarrow [X] = [Y]$

(ii) $\theta([X][Y]) = \theta([X]) \blacksquare \theta([Y])$

PROOF. (i) $\leftarrow$ is trivial. For $\rightarrow$, we prove with induction on n

$$\forall n\in\omega \forall b\in G(A_{c,e}) \forall b'\in G(A_{c,e})\ \varphi(b) = \varphi(b') \to b \le_{G(A_{c,e})} b' \ \&\ b' \le_{G(A_{c,e})} b.$$

Then if $\theta([X]) = \theta([Y])$, we have $X\equiv Y$ and thus $[X] = [Y]$. Clearly, this holds for $n=0$. Let $b, b'\in G_{n+1}(A_{c,e})$ and define

$$S(u) := \begin{cases} (\emptyset, u) \text{ if } u\in A_{c,e} \\ u \quad \text{otherwise.} \end{cases}$$

Then if $u\in A_{c,e}$ it follows from definition 3.2 that $\varphi(S(u)) = \varphi(u)$. Suppose $\varphi(b) = \varphi(b')$. Then also $\varphi(S(b)) = \varphi(S(b'))$. Hence $c(e^{-1}(\{\varphi(b)|b\in D_1\}), \varphi(d_1)) = = c(e^{-1}(\{\varphi(b)|b\in D_2\}), \varphi(d_2))$ where $S(b) = (D_1, d_1)$ and $S(b') = (D_2, d_2)$. So $\{\varphi(b)|b\in D_1\} = \{\varphi(b)|b\in D_2\}$ and $\varphi(d_1) = \varphi(d_2)$ and from the induction hypothesis it follows that $D_1 \le D_2$ and $d_1 \le_{G(A_{c,e})} d_2, d_2 \le_{G(A_{c,e})} d_1$. Hence

$$S(b) \le_{G(A_{c',e})} S(b') \text{ and } S(b') \le_{G(A_{c',e})} S(b).$$

Thus

$$b \le_{G(A_{c,e})} b' \text{ and } b' \le_{G(A_{c,e})} b.$$

(ii) By the proof of theorem 3.1 we have

$$\bigcup\{Z|Z\equiv XY\} = \bigcup\{Z|Z\equiv X\} \bullet \bigcup\{Z|Z\equiv Y\} =$$

$$= \{b|\exists B\subseteq \bigcup\{Z|Z\equiv Y\}(B,b)\in\bigcup\{Z|Z\equiv X\}\}.$$

Hence

$$\theta([X][Y]) = \theta([XY]) = \{\varphi(b) | b \in \cup\{Z | Z \equiv XY\}\} =$$
$$= \{\varphi(b) | \exists B \subseteq \cup\{Z | Z \equiv Y\}(B, b) \in \cup\{Z | Z \equiv X\}\} = \{\varphi(b) | \exists \text{ finite}$$
$$B \subseteq P(\omega)(B \subseteq \theta([Y]) \ \& \ c(e^{-1}(B), \varphi(b)) \in \theta([X]))\} =$$
$$= \{m \in \omega | \exists n \in \omega(e(n) \subseteq \theta([Y]) \ \& \ c(n, m) \in \theta([X]))\} = \theta([X]) \blacksquare \theta([Y]).$$

□

3.4 THEOREM. For all bijections $c : \omega^2 \to \omega$, $e : \omega \to \{X \in P(\omega) | X \text{ finite}\}$

$$(P(G(A_{c,e}))/\equiv, *) \hookrightarrow P\omega^{c,e}. \quad \square$$

Thus if $A_{c,e} \neq \emptyset$, i.e. if for some $n \in \omega$ $c(e^{-1}(\emptyset), n) = n$ then $P\omega^{c,e}$ has a substructure which can be made into an extensional *ca*.

EXAMPLES. Let e be the standard coding of finite subsets of ω defined by

$$e(n) = \{k_0, \ldots, k_{m-1}\} \text{ with } k_0 < \ldots < k_{m-1} \leftrightarrow n = 2^{k_0} + \ldots + 2^{k_{m-1}}$$

Then $e(\emptyset) = 0$. Consider the two codings of pairs c and c' given by

$$c(n, m) = \tfrac{1}{2}(n + m)(n + m + 1) + m$$

$$c'(n, m) = \tfrac{1}{2}(n + m)(n + m + 1) + n$$

Then $A_{c,e} = \{0\}$ and $A_{c',e} = \{0, 1\}$. By theorem 2.5 $M(A_{c,e}) \not\cong M(A_{c',e})$. Hence $P\omega^{c,e}$ and $P\omega^{c',e}$ contain non-isomorphic extensional *ca*'s.

ACKNOWLEDGEMENTS

I wish to thank Henk Barendregt for helpful discussions and encouragement. I am especially grateful to A.S. Troelstra for his comments and suggestions during the preparation of this paper.

REFERENCES

[B] Barendregt, H.P. – The Lambda Calculus, its Syntax and Semantics, North-Holland, Amsterdam, 1984

[E] Engeler, E. – Algebras and Combinators, Algebra Universalis, **13**, 389–392 (1981).

[S1] Scott, D.S. – Models for the λ-Calculus, Manuscript, 53 pp., (1969).

[S2] Scott, D.S. – Continuous Lattices, E. Lawvere, (ed.); Toposes, Algebraic Geometry and Logic; LNM **274**, Springer-Verlag; Berlin, 97–136 (1972).

[S3] Scott, D.S. – Data types as lattices, SIAM J. Comp., **5** (3); 522–587 (1976).